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SOLUTIONS OF EXERCISES.

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SOLVE the cubic $9x^3 + 9x = 2$.

SOLUTION.

By Cardan's formula

$$\alpha = \tfrac{1}{3}(\sqrt[3]{9} - \sqrt[3]{3}), \quad \beta = \tfrac{1}{3}(\omega\sqrt[3]{9} - \omega^2\sqrt[3]{3}), \quad \gamma = \tfrac{1}{3}(\omega^2\sqrt[3]{9} - \omega\sqrt[3]{3}),$$

in which ω, ω^2 are the imaginary cube roots of unity.

[*Jesse Pawling, Jr.*]

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INTEGRATE the differential

$$\frac{\sin\theta\cos\theta\,d\theta}{\sin^4\theta + \cos^4\theta}.$$

SOLUTION.

Put the expression in terms of 2θ and let $\cos 2\theta = x$, then

$$\int\frac{\sin\theta\cos\theta\,d\theta}{\sin^4\theta + \cos^4\theta} = -\tfrac{1}{2}\int\frac{dx}{1+x^2} = \tfrac{1}{2}\cot^{-1}\cos 2\theta.$$

[*Jesse Pawling, Jr.*]

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SHOW that the equation

$$x^3 + y^3 + z^3 - 3xyz = a^3$$

represents a surface of revolution and find its axis. [*Geo. R. Dean.*]

SOLUTION II.

The equation may be written

$$(x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz) = a^3,$$

or

$$(x + y + z)[3(x^2 + y^2 + z^2) - (x + y + z)^2] = 2a^3,$$

or

$$(x + y + z)^3 - 3(x^2 + y^2 + z^2)(x + y + z) = -2a^3.$$

Solving for $(x + y + z)$ we have

$$(x + y + z) = \varphi(x^2 + y^2 + z^2),$$

the general equation of a surface of revolution whose axis passes through the origin and has the equation $x = y = z$. [*Geo. R. Dean.*]

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On the axis of x take any point x_m, and equidistant from it on either side select points a and b. Draw verticals through a and b, also on the circle on ab as diameter. With a and b as centers and radius equal to a side of the inscribed square draw arcs cutting the verticals at a and b in a' and b'. The straight lines $a'x_m$ and $b'x_m$ cut the circle in points whose abscissæ are x_1 and x_2, respectively.

Show that if any cubic of the form

$$y = x^3 + ax^2 + \beta x + \gamma$$

be described passing through any two points x_1y_1 and x_2y_2 on the verticals at a and b respectively, then the area included between the curve, the verticals at a and b, and the x-axis, is constant and equal to

$$\tfrac{1}{2}(y_1 + y_2)(b - a). \qquad \text{[W. H. Echols.]}$$

SOLUTION.

After Gauss, the area of the curve

$$y = x^3 + ax^2 + bx + c,$$

taken between the limits $x = a$, $x = b$, is

$$\int_a^b \frac{1}{\beta - \alpha} \begin{vmatrix} 0 & 1 & x \\ y_\beta & 1 & \beta \\ y_\alpha & 1 & \alpha \end{vmatrix} dx,$$

wherein, after Jacobi, α and β are the roots of

$$\left[\frac{d}{dx}\right]^2 (x - a)^2 (x - b)^2 = 0,$$

or respectively are

$$\tfrac{1}{2}(a + b) \mp \frac{1}{2\sqrt{3}}(b - a).*$$

Very evidently these values of α and β satisfy the geometrical construction indicated for locating the ordinates y_1 and y_2.

The above integral effected, regard being had for the values of α and β, we obtain very readily for the area desired

$$\tfrac{1}{2}(b - a)(y_1 + y_2). \qquad \text{[W. H. Echols.]}$$

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Solve the equation

$$\frac{d^2y}{dx^2} - y = e^{\frac{1}{2}x^2}. \qquad \text{[Geo. R. Dean.]}$$

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* See Boole's Calculus of Finite Differences, 2d Ed., p. 52.

SOLUTION.

Writing the equation in the form $(D^2 - 1)\, y = e^{\frac{1}{2}x^2}$, we obtain

$$y = \frac{1}{D^2 - 1}\, e^{\frac{1}{2}x^2} = \frac{1}{2} \cdot \frac{1}{D-1}\, e^{\frac{1}{2}x^2} - \frac{1}{2} \cdot \frac{1}{D+1}\, e^{\frac{1}{2}x^2}.$$

Substituting z for $\frac{1}{2}x^2$, we get

$$\frac{1}{D_x - 1}\, e^{\frac{1}{2}x^2} = \frac{1}{D_z - \frac{1}{\sqrt{2z}}} \cdot \frac{e^z}{\sqrt{2z}} = e^{\sqrt{2z}} \int e^{z - \sqrt{2z}} \frac{dz}{\sqrt{2z}} + c_1 e^{\sqrt{2z}},$$

$$\frac{1}{D_x + 1}\, e^{\frac{1}{2}x^2} = \frac{1}{D_z + \frac{1}{\sqrt{2z}}} \cdot \frac{e^z}{\sqrt{2z}} = e^{-\sqrt{2z}} \int e^{z + \sqrt{2z}} \frac{dz}{\sqrt{2z}} + c_2 e^{-\sqrt{2z}}.$$

Therefore,

$$y = c\left[e^x \int e^{\frac{1}{2}x^2 - x}\, dx - e^{-x} \int e^{\frac{1}{2}x^2 + x} dx\right] + c_1 e^x + c_2 e^{-x}.$$

[*Geo. R. Dean.*]

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PROVE that the ratio of

$$\begin{vmatrix} \sigma_1(2u_1), & \sigma_1(2u_2), & \sigma_1(2u_3) \\ \sigma_2(2u_1), & \sigma_2(2u_2), & \sigma_2(2u_3) \\ \sigma_3(2u_1), & \sigma_3(2u_2), & \sigma_3(2u_3) \end{vmatrix}$$

to

$$\sigma(u_2 + u_3)\, \sigma(u_3 + u_1)\, \sigma(u_1 + u_2)\, \sigma(u_2 - u_3)\, \sigma(u_3 - u_1)\, \sigma(u_1 - u_2)$$

is independent of the arguments u_λ; and that its value is

$$4\,(e_2 - e_3)\,(e_3 - e_1)\,(e_1 - e_2)\,;$$

the notation being that of Weierstrass. [*Frank Morley.*]

SOLUTION.

We know that $(\sigma_\lambda u/\sigma u)^2$ is an elliptic function of u (Harkness and Morley's Theory of Functions, p. 309, formula 35), and that $\sigma_\lambda(2u)$ is a quadratic form in $\sigma_\lambda{}^2 u$, $\sigma_\mu{}^2 u$, $\sigma_\nu{}^2 u$ (p. 316). Hence $\sigma_\lambda(2u)/\sigma^4 u$ is an elliptic function with periods $2\omega_\lambda$; and the determinant is therefore made an elliptic function of each argument by dividing it by $(\sigma u_1\, \sigma u_2\, \sigma u_3)^4$. But the same process renders the denominator of the ratio elliptic (see p. 307, formula 30). Hence the ratio is an elliptic function of each argument.

Regarded as a function of u_λ, the ratio can be ∞ only when $u_\lambda = \pm\, u_\mu$ or $\pm\, u_\nu$; for these are the zeros of the denominator. But any one of these conditions makes two columns of the determinant identical. Thus, observing that these zeros are only of the first order, we see that the ratio is nowhere infinite. Hence our elliptic function of u_λ has no poles, and therefore (p. 288) it is independent of u_λ. Therefore it is a constant.

To determine this constant let $u_\lambda = \omega_\lambda$ where $\lambda = 1, 2, 3$. The determinant is (p. 312, formula 43)

$$\begin{vmatrix} -1, & 1, & 1 \\ 1, & -1, & 1 \\ 1, & 1, & -1 \end{vmatrix} \exp 2\Sigma\eta_\lambda\omega_\lambda ,$$

the periods being such that $\Sigma\omega_\lambda = 0$, $\Sigma\eta_\lambda = 0$.

The denominator is (p. 312, formula 44)

$$-(e_2 - e_3)(e_3 - e_1)(e_1 - e_2)(\sigma\omega_1\,\sigma\omega_2\,\sigma\omega_3)^4 .$$

But (p. 311, formula 41)

$$\exp 2\Sigma\eta_\lambda\omega_\lambda = -[(e_2 - e_3)(e_3 - e_1)(e_1 - e_2)]^2(\sigma\omega_1\,\sigma\omega_2\,\sigma\omega_3)^4 .$$

Hence the constant ratio is

$$4(e_2 - e_3)(e_3 - e_1)(e_1 - e_2) .$$ [*Frank Morley.*]

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WHEN the bilinear invariant of two binary n-ics is zero, we say that the n-ics are *apolar*. When also the n-ics coincide we say that either is self-apolar. And we may apply the same adjectives to the sets of n points (or n-*ads*) which represent the zeros of the n-ics. Any odd set of points is, we know, self-apolar (Salmon's Higher Algebra, § 153). Prove that an even set is self-apolar when the first polar of any point of the set, with regard to the rest, is self-apolar. [*Frank Morley.*]

SOLUTION.

Writing the equation which determines the n-ad, when $n = 2h$, in the shape

$$s_0x^{2h} - s_1x^{2h-1} + s_2x^{2h-2} - \ldots + s_{2h} = 0 , \qquad (1)$$

the condition of self-apolarity is

$$(2h)!\, s_0s_{2h} - 1!\,(2h-1)!\, s_1s_{2h-1} + \ldots + (-)^h s_h^2\, h!^2/2 = 0 . \qquad (2)$$

Take the selected point as origin, so that $s_{2h} = 0$; the first polar of the origin with regard to all the points is

$$s_1x^{2h-1} - 2s_2x^{2h-2} + \ldots + (2h-1)\,s_{2h-1}x = 0 ;$$

and when we remove the factor x, this gives the first polar of the origin as to the remaining $2h - 1$ points. Hence, if these $2h - 2$ points be given by

$$\sigma_0 x^{2h-2} - \sigma_1 x^{2h-1} + \ldots + \sigma_{2h-2} = 0\,,$$

we have

$$s_1\sigma_1 = 2s_2\sigma_0\,,\quad s_1\sigma_2 = 3s_3\sigma_0\,,\quad \ldots,\quad s_1\sigma_{2h-2} = (2h-1)\, s_{2h-1}\,\sigma_0\,;$$

therefore (2), when expressed in terms of σ, is

$$(2h-2)!\,\sigma_0\,\sigma_{2h-2} - (2h-3)!\,\sigma_1\,\sigma_{2h-3} + \ldots + (-)^{h-1}\sigma_{h-1}{}^2\,(h-1)!^2/2 = 0\,,$$

a relation which differs from (2) only in the change from h to $h - 1$. Hence the theorem. A system of two points is self-apolar when the points coincide; a system of 4 points which is self-apolar is also "equi-anharmonic."

[*Frank Morley.*]

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Prove that in an ellipse the area of the triangle formed by the tangents at the points whose eccentric angles are α, β, γ, respectively, is

$$A = ab \tan \tfrac{1}{2}(\alpha - \beta) \tan \tfrac{1}{2}(\beta - \gamma) \tan \tfrac{1}{2}(\gamma - \alpha)\,.$$

[*F. P. Matz.*]

This exercise is given by Salmon (Conic Sections, 6th ed., p. 220).

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Show without resorting to the Differential Calculus that the two parabolas whose equations are $y^2 = ax$ and $x^2 = by$ intersect at an angle

$$\theta = \tan^{-1}[3a^{\frac{1}{3}}b^{\frac{1}{3}}/2(a^{\frac{2}{3}} + b^{\frac{2}{3}})]\,.$$

[*F. P. Matz.*]

SOLUTION.

At the point of intersection

$$x = a^{\frac{1}{3}}b^{\frac{2}{3}}\,,\quad y = a^{\frac{2}{3}}b^{\frac{1}{3}}\,.$$

Since the subtangent to the parabola is double the abscissa, the inclinations of the tangents at this point to the x-axis are

$$\tan\alpha = \frac{b^{\frac{1}{3}}}{2a^{\frac{1}{3}}}\,,\qquad \tan\beta = \frac{2b^{\frac{1}{3}}}{a^{\frac{1}{3}}}\,;$$

$$\therefore\ \tan\beta - \tan\alpha = \frac{3b^{\frac{1}{3}}}{2a^{\frac{1}{3}}}\,,\qquad 1 + \tan\beta\tan\alpha = 1 + \frac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}\,;$$

and therefore

$$\tan\theta = 3a^{\frac{1}{3}}b^{\frac{1}{3}}/[2(a^{\frac{2}{3}} + b^{\frac{2}{3}})]\,.$$

[*P. H. Philbrick.*]

Solved also by H. Y. Benedict, Geo. R. Dean, William Hoover, G. B. M. Zerr, W. O. Whitescarver, and Jesse Pawling, Jr.

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IF the hodograph of a curve described under constant acceleration be a parabola in which the radius-vectors are drawn from the focus, the intrinsic equation of the curve described is

$$s = c \int \sec^5 \tfrac{1}{2}\varphi \, d\varphi \, . \qquad \text{[F. P. Matz.]}$$

SOLUTION.

The equation of the hodograph may be written $r = k \sec^2 \frac{1}{2}\varphi$, where k is a constant. But the radius vector in the hodograph equals the velocity in orbit, or $\frac{ds}{dt} = k \sec^2 \frac{1}{2}\varphi$. The velocity in the hodograph

$$= \frac{d\varphi}{dt}\sqrt{r^2 + \left[\frac{dr}{d\varphi}\right]^2} = k \sec^3 \tfrac{1}{2}\varphi \frac{d\varphi}{dt}$$

= the acceleration in the orbit $= f$, a constant. Eliminating dt and writing c for $\frac{k^2}{f}$, there results $ds = c \sec^5 \frac{1}{2}\varphi \, d\varphi$. [*H. Y. Benedict.*]

Solved also by Geo. R. Dean, William Hoover, and G. B. M. Zerr.

376

AT each end of a horizontal base-line $2a$ the angle of elevation of a monument is θ, and at the middle point of this base-line the angle of elevation is φ. Prove that the elevation of the monument is

$$H = \frac{a \sin\varphi \sin\theta}{\sqrt{[\sin(\varphi + \theta)\sin(\varphi - \theta)]}} \, . \qquad \text{[F. P. Matz.]}$$

SOLUTION.

The projection of the line from the end of the base to the top of the monument is $H \cot\theta$, and the projection of the line from the middle of the base is $H \cot\varphi$. The latter is evidently perpendicular to the base.

$$\therefore \; H^2 \cot^2\theta - H^2 \cot^2\varphi = a^2 \, , \qquad (1)$$

$$H^2 = \frac{a^2 \sin^2\varphi \sin^2\theta}{\sin^2\varphi \cos^2\theta - \cos^2\varphi \sin^2\theta} \, , \qquad (2)$$

$$H = \frac{a \sin\varphi \sin\theta}{\sqrt{\sin(\varphi + \theta)\sin(\varphi - \theta)}} \, .$$

[*W. O. Whitescarver.*]

Solved also by H. Y. Benedict, P. H. Philbrick, G. B. M. Zerr, and Jesse Pawling, Jr.

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PROVE that the rectangle under the focal distances of the origin in the conic represented by the equation $ax^2 + 2hxy + by^2 = 2y$, is $R = 1/(ab - h^2)$.

[*F. P. Matz.*]

SOLUTION.

The foci are given by

$$(Cx + h)^2 = \tfrac{1}{2}\Delta(R + a - b), \quad (Cy - a)^2 = \tfrac{1}{2}\Delta(R + b - a),$$

(See Salmon's Conic Sections, p. 239, exercise and Art. 151) where $C = ab - h^2$, $R^2 = 4h^2 + (a - b)^2$, and Δ, for this exercise, $= -a$.

$$\therefore\ (Cx + h)^2 = \tfrac{1}{2}a(b - a - R) = A^2,$$

$$(Cy - a)^2 = \tfrac{1}{2}a(a - b - R) = B^2;$$

$$\therefore\ x = \frac{-h \pm A}{C}, \quad y = \frac{a \mp B}{C};$$

$$\therefore\ R = \frac{1}{C^2}\sqrt{\{(A + h)^2 + (a + B)^2\}\{(A - h)^2 + (a - B)^2\}}$$

$$= \frac{1}{C} = \frac{1}{ab - h^2}.$$

[*G. B. M. Zerr.*]

Also solved by William Hoover and W. O. Whitescarver.

EXERCISES.

379

FIND the radius of a circle circumscribing the three tangent-circles of radii a, b, and c, respectively. [*F. P. Matz.*]

380

THE sides of a variable rectangle pass through four fixed points. Find the position of the rectangle and its dimensions when its area is a maximum.

[*Geo. R. Dean.*]

381

FROM a point in the circumference of a circle of radius R as centre is described the external arc of a circle of radius r. Determine r so that the area of the lune shall equal that of the original circle. [*W. M. Thornton.*]